(42); (62) is the general solution.* A few particular results are given. If $r=1$, the right member of (62) becomes e, as it should, since $\kappa_1=1$.

If $r=2$, $\kappa_1+2\kappa_2=2$,	$\kappa_1=2$	1	Sum
	$\kappa_2=1$	1	$2e$
		—	
$r=3$, $\kappa_1+2\kappa_2+3\kappa_3=3$,	$\kappa_1=3$	1	
	$\kappa_1=1$, $\kappa_2=2$	3	
	$\kappa_3=1$	1	$5e$
		—	
$r=4$, $\kappa_1+2\kappa_2+3\kappa_3+4\kappa_4=4$,	$\kappa_1=4$	1	
	$\kappa_1=2$, $\kappa_2=1$	6	
	$\kappa_1=1$, $\kappa_3=1$	4	
	$\kappa_2=2$	3	
	$\kappa_4=1$	1	$15e$

1. THE MONOMIAL SYMMETRIC FUNCTIONS IN TERMS OF THE t'S.

Weight 1.

	t_1
(1)	1

Weight 2.

	t_1^2	t_2
(2)	−1	2
(1^2)	1	−1

Weight 3.

	t_1^3	t_1t_2	t_3
(3)	1	−3	3
(21)	−2	5	−3
(1^3)	1	−2	1

Weight 4.

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
(4)	−1	4	−2	−4	4
(31)	2	−7	2	7	−4
(2^2)	1	−4	3	2	−2
(21^2)	−3	10	−4	−7	4
(1^4)	1	−3	1	2	−1

Weight 5.

	t_1^5	$t_1^3t_2$	$t_1t_2^2$	$t_1^2t_3$	t_2t_3	t_1t_4	t_5
(5)	1	−5	15	5	−5	−5	5
(41)	−2	9	−7	−9	5	9	−5
(32)	−2	10	−11	−8	11	5	−5
(31^2)	3	−13	10	12	−8	−9	5
(2^21)	3	−14	14	10	−11	−7	5
(21^3)	−4	17	−14	−13	10	9	−5
(1^5)	1	−4	3	3	−2	−2	1

These tables are symmetrical with respect to the secondary diagonal. The sum of the numbers of the last column is unity, while that of the other columns is zero, because the sum of the monomial symmetric functions in the vertical column at the left of the table is, by definition, the complete symmetric function of weight w, t_w.

Multiplying the elements of the inverse table columnwise by those of the lines of the direct table (linewise), and so that the order is the same in both tables, and summing, we get zero, except when line and column belong to the same partition. Then the sum is unity. The reason for this is that the identity which so arises requires it. This property is true of all direct and inverse tables of the kind given in this paper. It is illustrated by the following arrangement of the direct and inverse tables for weight four.

†Cf. The American Mathematical Monthly, Vol. 9 (1902), pages 11 and 12. C. Smith's Treatise on Algebra, example 18, page 387. Loney's Trigonometry, Part II, page 9, example 7.

Direct Table (read vertically).

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
(4)	−1	4	−2	−4	4
(31)	2	−7	2	7	−4
(2^2)	1	−4	3	2	−2
(21^2)	−3	10	−4	−7	4
(1^4)	1	−3	1	2	−1

Inverse Table (read vertically).

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
(4)	1	1	1	1	1
(31)	4	3	2	2	1
(2^2)	6	4	3	2	1
(21^2)	12	7	4	3	1
(1^4)	24	12	6	4	1

Here the sum of the products of the elements of any two lines or of any two columns of the two tables is zero unless the two lines or the two columns are similarly placed, when the sum of the products of their elements is equal to unity. If two determinants be formed of the two arrays of the direct and inverse table, it thus appears that the product of these two determinants is unity. Hence the determinant of each array is also equal to unity numerically, when each determinant is integral; in other cases these determinants are reciprocals of each other.

2. The t's in Terms of Monomial Symmetric Functions. Inverse Table.

Weight 1.

	(1)
t_1	1

Weight 2.

	(1^2)	(2)
t_2	1	1
t_1^2	2	1

Weight 3.

	(1^3)	(21)	(3)
t_3	1	1	1
t_1t_2	3	2	1
t_1^3	6	3	1

Weight 4.

	(1^4)	(21^2)	(2^2)	(31)	(4)
t_4	1	1	1	1	1
t_1t_3	4	3	2	2	1
t_2^2	6	4	3	2	1
$t_1^2t_2$	12	7	4	3	1
t_1^4	24	12	6	4	1

Weight 5.

	(1^5)	(21^3)	(2^21)	(31^2)	(32)	(41)	(5)
t_5	1	1	1	1	1	1	1
t_1t_4	5	4	3	3	2	2	1
t_2t_3	10	7	5	4	3	2	1
$t_3t_1^2$	20	13	8	7	4	3	1
$t_2^2t_1$	30	18	11	8	5	3	1
$t_1^3t_2$	60	33	18	13	7	4	1
t_1^5	120	60	30	20	10	5	1

In these tables the elements are symmetrical with respect to the secondary diagonal. The elements are all positive. The relations into which they enter with those of the direct table have already been mentioned.

3. The t's in Terms of the b's.

Weight 1.

	b_1
t_1	−1

Weight 2.

	b_1^2	b_2
t_2	1	−1
t_1^2	1	

Weight 3.

	b_1^3	b_1b_2	b_3
t_3	−1	2	−1
t_1t_2	−1	1	
t_1^3	−1		

Weight 4.

	b_1^4	$b_1^2b_2$	b_2^2	b_1b_3	b_4
t_4	1	−3	1	2	−1
t_1t_3	1	−2	0	1	
t_2^2	1	−2	1		
$t_2t_1^2$	1	−1			
t_1^4	1				

Weight 5.

	b_1^5	$b_1^3b_2$	$b_1b_2^2$	$b_1^2b_3$	b_2b_3	b_1b_4	b_5
t_5	−1	4	−3	−3	2	2	−1
t_1t_4	−1	3	−1	−2	0	1	
t_2t_3	−1	3	−2	−1	1		
$t_1^2t_3$	−1	2	0	−1			
$t_1t_2^2$	−1	2	−1				
$t_1^3t_2$	−1	1					
t_1^5	−1						

In these tables the elements are contained within a right triangle. Symmetry is not present. . Elements in the same column have the same sign. The sum of all the elements in a horizontal line (the last one excepted) vanishes, as it must according to (28). The sum of all the elements is $(-1)^w$ for the same reason. The relations to the inverse table are seen by writing the inverse as in 1. to read vertically.

4. The b's in Terms of the t's. (Inverse of 3.)

In 3, interchange b and t, and we have the table, on account of (23) as worked out in (26) and (27).

5. The s's in Terms of the t's. (The s's in Terms of the b's.)

Weight 1.

	t_1
s_1	1

Weight 2.

	t_1^2	t_2
s_2	−1	2
s_1^2	1	

Weight 3.

	t_1^3	t_1t_2	t_3
s_3	1	−3	3
s_1s_2	−1	2	
s_1^3	1		

Weight 4.

	t_1^4	$t_1^2t_2$	t_2^2	t_1t_3	t_4
s_4	−1	4	−2	−4	4
s_1s_3	1	−3	0	3	
s_2^2	1	−4	4		
$s_1^2s_2$	−1	2			
s_1^4	1				

Weight 5.

	t_1^5	$t_2t_1^3$	$t_2^2t_1$	$t_3t_1^2$	t_3t_2	t_1t_4	t_5
s_5	1	−5	5	5	−5	−5	5
s_4s_1	−1	4	−2	−4	0	4	
s_3s_2	−1	5	−6	−3	6		
$s_3s_1^2$	1	−3	0	3			
$s_2^2s_1$	1	−4	4				
$s_2s_1^3$	−1	2					
s_1^5	1						

Here the elements are contained in a right triangle. The sum of the elements in a horizontal line is 1, as it should be by (32) and (34).

If s is changed into $-s$, and t into b, these tables then express s's in terms of b's, on account of (35) and (36) as before observed. This amounts to changing t into b, and changing or not changing all the signs of the elements in a horizontal line according as the product of the s's at the left of the line is of odd or of even degree. And the sums of such collinear elements are accordingly ∓ 1.

6. THE t'S IN TERMS OF THE s'S. (THE b'S IN TERMS OF THE s'S.) (Inverse of 5.)

Weight 1.

	s_1
t_1	1

Weight 2.

	s_1^2	s_2
t_2	$\frac{1}{2}$	$\frac{1}{2}$
t_1^2	1	

Weight 3.

	s_1^3	$s_1 s_2$	s_3
t_3	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$
$t_1 t_2$	$\frac{1}{2}$	$\frac{1}{2}$	
t_1^3	1		

Weight 4.

	s_1^4	$s_1^2 s_2$	s_2^2	$s_1 s_3$	s_4
t_4	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	$\frac{1}{4}$
$t_1 t_3$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{3}$	
t_2^2	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$		
$t_1^2 t_2$	$\frac{1}{2}$	$\frac{1}{2}$			
t_1^4	1				

Weight 5.

	s_1^5	$s_1^3 s_2$	$s_1 s_2^2$	$s_1^2 s_3$	$s_2 s_3$	$s_1 s_4$	s_5
t_5	$\frac{1}{120}$	$\frac{1}{12}$	$\frac{1}{8}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{5}$
$t_1 t_4$	$\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{3}$	0	$\frac{1}{4}$	
$t_2 t_3$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$		
$t_1^2 t_3$	$\frac{1}{6}$	$\frac{1}{2}$	0	$\frac{1}{3}$			
$t_1 t_2^2$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$				
$t_1^3 t_2$	$\frac{1}{2}$	$\frac{1}{2}$					
t_1^5	1						

The sum of the horizontally collinear elements is 1, as it should be by (41).

If s is changed into $-s$, *i. e.*, if the signs of the elements of the columns are or are not changed according as the product of the s's heading the column is of odd or of even degree, and the t's are replaced by b's, we have then the table for the b's expressed in terms of the s's. In this case the sum of horizontally collinear elements, except for the last line, is zero, by (42) and (43).

TWO INFINITE SYSTEMS OF GROUPS GENERATED BY TWO OPERATORS OF ORDER FOUR.

By PROFESSOR G. A. MILLER. Stanford University.

If a, b, c represent respectively the orders of two operators and that of their product, the group generated by these operators is completely defined by the numbers a, b, c only in the following cases: (1) When one of these three numbers is unity; (2) When two of them are equal to 2 and the third greater than 1; (3) When they have one of the following three sets of values: (2, 3, 3), (2, 3, 4), (2, 3, 5). The corresponding groups are respectively cyclic, dihedral rotation, tetrahedral, octahedral, and icosahedral.*

The abstract definitions and the laws of combination of these groups are so simple that they furnish very interesting examples in the logic of algebra. We proceed to consider two other very elementary systems of groups. Let s_1, s_2, represent two non-commutative operators of order four such that

*Burnside, Theory of groups of finite order, 1897, p. 291; Miller, *Bulletin of American Mathematical Society*, Vol. 7, 1901, p. 424; *American Journal of Mathematics*, Vol. 24, 1902, p. 96.

$$s_1^2 s_2^2 = 1.$$

By multiplying on the right and the left by s_1 and s_2 respectively there results

$$s_1^3 s_2^3 = s_1 s_2 = s_1^{-1} s_2^{-1} = (s_2 s_1)^{-1}.$$

Since $s_2^{-1}(s_2 s_1)s_2 = s_1 s_2 = (s_2 s_1)^{-1}$, it follows that s_2 transforms $s_2 s_1$ into its inverse. The order (g) of the group (G) generated by s_1 and s_2 is therefore equal to twice the order of $s_1 s_2$ whenever the group (H_1) generated by $s_1 s_2$ includes $s_1^2 = s_2^2$. When this condition is not satisfied $g = 4h$, h being the order of H_1.

In the former case G is composed of the cyclic group H_1 of order $g/2$, and $g/2$ operators of order 4 which have a common square.* Each of these operators of order 4 transforms each operator of H_1 into its inverse. As s_1 and s_2 can be so selected that $g/2$ is an arbitrary even number, this infinite system of groups is composed of just one group of each order $4k$, $k = 2, 3, \ldots\ldots, \infty$. This system is interesting also because *it includes all possible non-cyclic groups of order p^m, p being a prime, which have only one subgroup of order p^s, $0 < s < m$.*†

In the latter case G includes the direct product (H_2) of an operator of order two (s_1^2) and the cyclic group of order $g/4$ generated by $s_1 s_2$. The remaining $g/2$ operators of G are of order four, have a common square, and transform each operator of H_2 into its inverse. This infinite system is composed of just one group of each order $4k$, $k = 3, 4, \ldots\ldots, \infty$. The value $k = 2$ is excluded by the fact that two non-commutative operators of order 4 cannot satisfy both the conditions

$$s_1^2 s_2^2 = 1 \text{ and } (s_2 s_1)^2 = 1.$$

In fact, it results from the former that $s_1^3 s_2^3 = s_1 s_2$, and from the latter that $s_2 s_1 s_2 s_1 = 1$, or that $s_2 s_1 = s_1^3 s_2^3$. Hence $s_1 s_2 = s_2 s_1$. When k is odd, H_2 is cyclic.

Every group generated by two non-commutative operators of order four which have a common square must belong to one of the given systems. Hence such a group may be completely defined by saying that it is generated by two non-commutative operators of order 4 which have a common square and whose product is of order $h > 2$; when h is even it is necessary to add whether the product does or does not generate the square of one of the given generators.

*This statement follows directly from the fact that s_2 transforms each operator of H_1 into its inverse.
†Burnside, Theory of Groups, p. 75.

ON A CYLINDER THE INTERSECTION OF WHICH WITH A SPHERE WILL DEVELOP INTO AN ELLIPSE.*

By D. N. LEHMER, University of California.

1. In his *Mathematical Recreations* published about 1674 Van Etten describes a method of drawing an "oval" with a pair of compasses leaving the distance between the points of the instrument unaltered. His scheme was to stretch the drawing paper around a cylinder. The curve thus obtained is not an ellipse but a transcendental curve whose equation is

$$y^2+4b^2\sin^2\frac{x}{2b}=a^2,$$

where b is the radius of the cylinder and a the distance between the points of the compasses. The curve is clearly the developed intersection of a sphere of radius a with a cylinder of radius b, the center of the sphere being on the surface of the cylinder.

2. The question suggests itself to find the cylinder such that the curve as obtained above will develop into an ellipse. Take the origin at the center of the sphere and let the y-axis be an element of the cylinder. Let a plane parallel to the xz-plane cut out on the cylinder a curve of length s. Then the equation of the ellipse in question will be

$$\frac{y^2}{a^2}+\frac{s^2}{b^2}=1.$$

To find the nature of the cylinder we have then the equations

$$x^2+y^2+z^2=a^2 \quad \text{.....(1)},$$

$$\frac{y^2}{a^2}+\frac{s^2}{b^2}=1 \quad \text{.....(2)},$$

$$\frac{ds}{dx}=\sqrt{1+\frac{dz^2}{dx^2}} \quad \text{.....(3)}.$$

From these it is easy to derive the differential equation connecting x and z:

$$a^2(x^2+z^2)\left(1+\frac{dz^2}{dx^2}\right)=b^2\left(z\frac{dz}{dx}+x\right)^2.$$

Changing to polar coördinates and reducing,

*Read before the San Francisco Section of the American Mathematical Society.

$$\frac{d\rho}{d\theta} = \pm \frac{a\rho}{\sqrt{(b^2-a^2)}}, \text{ or } \kappa\rho = e^{\pm [a/\sqrt{(b^2-a^2)}]\theta}.$$

The cylinder has therefore for its right section a logarithmic spiral or rather two arranged in opposite directions.

3. This curious result may be stated in the following way. If an elliptical hole be cut out of a sheet of paper and the sheet wrapped about a sphere the diameter of which equals the minor axis of the ellipse the sheet wraps up into a cylinder a cross section of which is a logarithmic spiral. A model illustrating this relation between the ellipse and the spiral is easily constructed.

BERKELEY, *April, 1904.*

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

200. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

No matter what value x be given, the *numerical* value of the expression $(x+2)/(2x^2+3x+6)$ can never exceed $\frac{1}{3}$.

II. Solution by G. W. GREENWOOD, M.A., Professor of Mathematics and Astronomy in McKendree College, Lebanon, Ill.

We know that $y+1/y \not< 2$.

$\therefore z/2+2/z \not< 2$. $2z+8/z \not< 8$. $2z+8/z-5 \not< 3$. $\frac{2z^2-5z+8}{z} \not< 3$.

Now put $z=x+2$ and we get

$$\frac{2x^2+3x+6}{x+2} \not< 3; \text{ } i.\ e., \frac{x+2}{2x^2+3x+6} \not> \tfrac{1}{3}.$$

202. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Express in the form of radicals the roots of the equation

$$x^9+9mx^7+27m^2x^5+30m^3x^3+9mx+2r=0.$$

I. Solution by A. H. HOLMES, Brunswick, Maine.

Writing $x^3+3mx=y$ the equation becomes

$$y^3+3m^3y+2r=0 \ldots\ldots (1).$$

Let $s_1=[-r+\sqrt{(m^9+r^2)}]^{\frac{1}{3}}$, $s_2=[-r-\sqrt{(m^9+r^2)}]^{\frac{1}{3}}$, and the roots of (1) become $y_1=s_1+s_2$, $y_2=\frac{1}{2}(s_1+s_2)+\frac{1}{2}\sqrt{(-3)}(s_1-s_2)$, $y_3=-\frac{1}{2}(s_1+s_2)-\frac{1}{2}\sqrt{(-3)}(s_1-s_1)$.

$\therefore x^3+3mx=y_1$ or y_2 or y_3 ..... (2).

The nine roots of the three cubics (2) give the required solution. We let

$$s_3=\left(-\frac{y_1}{2}+\sqrt{(m^3+\tfrac{1}{4}y_1^2)}\right)^{\frac{1}{3}},\quad s_4=\left(-\frac{y_1}{2}-\sqrt{(m^3+\tfrac{1}{4}y_1^2)}\right)^{\frac{1}{3}}$$

and we denote the corresponding radicals when y_1 is replaced by y_2, by s_5, s_6; and when y_1 is replaced by y_3 by s_7, s_8. The nine roots of the given equation are:

$x_1=s_3+s_4$, $\quad x_4=s_5+s_6$,

$x_2=-\frac{1}{2}(s_3+s_4)+\frac{1}{2}\sqrt{(-3)}(s_3-s_4)$, $\quad x_5=-\frac{1}{2}(s_5+s_6)+\frac{1}{2}\sqrt{(-3)}(s_5-s_6)$,

$x_3=-\frac{1}{2}(s_3+s_4)-\frac{1}{2}\sqrt{(-3)}(s_1-s_4)$, $\quad x_6=-\frac{1}{2}(s_5+s_6)-\frac{1}{2}\sqrt{(-3)}(s_5-s_6)$,

$x_7=s_7+s_8$,

$x_8=-\frac{1}{2}(s_7+s_8)+\frac{1}{2}\sqrt{(-3)}(s_7-s_8)$,

$x_9=-\frac{1}{2}(s_7+s_8)-\frac{1}{2}\sqrt{(-3)}(s_7-s_8)$.

Similarly solved by Elmer Schuyler, and J. Scheffer.

II. Solution by the PROPOSER.

Let $x=y+z$. Then $x^9+9mx^7+27m^2x^5+30m^3x^3+9m^4x+2r=0$, becomes

$y^9+z^9+(yz+m)[9(y^7+z^7)+(36yz+27m)(y^5+z^5)+(84y^2z^2+105myz+30m^2)(y^3+z^3)+(126y^3z^3+189my^2z^2+81m^2yz+9m^3)(y+z)+2r=0$.

$\therefore y^9+z^9=-2r$ and $yz=-m$.

$\therefore y^{18}+2ry^9=m^9$. $\quad \therefore y^9=-r\pm\sqrt{(r^2+m^9)}$.

Let $a^9=-r+\sqrt{(r^2+m^9)}$, $b^9=-r-\sqrt{(r^2+m^9)}$.

Let $w=\dfrac{-1+\sqrt{(-3)}}{2}=\beta^3$.

Then the nine roots are $a+b$, $\beta a+\beta^8 b$, $\beta^2 a+\beta^7 b$, $\beta^3 a+\beta^6 b$, $\beta^4 a+\beta^5 b$, $\beta^5 a+\beta^4 b$, $\beta^6 a+\beta^3 b$, $\beta^7 a+\beta^2 b$, $\beta^8 a+\beta b$.

Also solved by L. E. Newcomb, and F. D. Posey.

203. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

The sum of a certain pair of roots of $x^4+ax^3+\left(\dfrac{2b}{a}+\dfrac{a^2}{4}\right)x^2+bx+c=0$ is equal to the sum of the remaining pair.

I. Solution by J. SCHEFFER, Hagerstown, Md.

The problem reduces to the problem of finding the value of m in the equation $a^4+ax^3+mx^2+bx+c=0$, subject to the condition that the sum of two roots equals the sum of the two others. Denoting the roots by α, β, γ, δ, we have $\alpha+\beta=\gamma+\delta=-\frac{1}{2}a$. Since $\alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta=-b$, we have by factoring,

$$\alpha\beta(\gamma+\delta)+(\alpha+\beta)\gamma\delta=-b, \text{ or } (\alpha\beta+\gamma\delta)\tfrac{1}{2}a=b.$$

$\therefore \alpha\beta+\gamma\delta=2b/a$. Since $\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta=m$, we have $(\alpha\beta+\gamma\delta)+(\alpha+\beta)(\gamma+\delta)=m$, or $(\alpha\beta+\gamma\delta)+(\alpha+\beta)^2=m$, or $2b/a+\frac{1}{4}a^2=m$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Place $x^4+ax^3+(2b/a+a^2/4)x^2+bx+c=(x^2+kx+m)(x^2+kx+n)$ where k is the sum of two of the roots. Since the sum of two roots equals the sum of the other two, k must be the same in both factors. Equating like powers of x we get $2k=a$, $k^2+m+n=2b/a+a^2/4$, $k(m+n)=b$, $mn=c$.

$\therefore k=a/2$, $m+n=2b/a$, $mn=c$, etc. The roots are now easily found—

$$m=\frac{1}{a}[b\pm\sqrt{(b^2-a^2c)}],\quad n=\frac{1}{a}[\mp\sqrt{(b^2-a^2c)}].$$

$$\therefore x=-\tfrac{1}{2}[k\mp\sqrt{(k^2-4m)}]=-\tfrac{1}{2}\left(\frac{a}{2}\mp\sqrt{\frac{a^2}{4}-\frac{4}{b}[b\pm\sqrt{(b^2-a^2c)}]}\right).$$

$$x=-\tfrac{1}{2}[k\mp\sqrt{(k^2-4n)}]=-\tfrac{1}{2}\left(\frac{a}{2}\mp\sqrt{\frac{a^2}{4}-\frac{4}{a}[b\mp\sqrt{(b^2-a^2c)}]}\right).$$

Analogously solved by F. D. Posey.

III. Solution by A. H. HOLMES, Brunswick, Maine, and L. E. NEWCOMB, Los Gatos, California.

Transposing c and adding b^2/a^2 to both sides, the roots of the equation are easily found to be

$$x_1=-\tfrac{1}{2}\left[\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{4}{a}[b+\sqrt{(b^2-4ac)}]}\right]$$

$$x_2=-\tfrac{1}{2}\left[\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{4}{a}[b+\sqrt{(b^2-4ac)}]}\right]$$

$$x_3=-\tfrac{1}{2}\left[\frac{a}{2}+\sqrt{\frac{a^2}{4}-\frac{4}{a}[b-\sqrt{(b^2-4ac)}]}\right]$$

$$x_4=-\tfrac{1}{2}\left[\frac{a}{2}-\sqrt{\frac{a^2}{4}-\frac{4}{a}[b-\sqrt{(b^2-4ac)}]}\right]$$

Evidently, $x_1+x_2=x_3+x_4=-\frac{1}{2}a$.

Also solved by Elmer Schuyler.

204. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

If α, β, γ be the roots of the cubic equation $x^3+qx+r=0$, prove that $3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4$.

I. Solution by PHILIP GRABER, Ph. B., M. S., Akron, Ohio.

Since $\alpha+\beta+\gamma=0$ we may write $\alpha=-2s$, $\beta=s+t$, $\gamma=s-t$. Then

$\Sigma\alpha^2=6s^2+2t^2$.....(1), $\Sigma\alpha^3=-6s^3+6st^2$.....(2),

$\Sigma\alpha^4=18s^4+12s^2t^2+2t^4$.....(3), and $\Sigma\alpha^5=-30s^5+20s^3t^2+10st^4$.....(4).

(1) multiplied by (4) gives $\Sigma\alpha^2\Sigma\alpha^5=-180s^7+60s^5t^2+100s^3t^4+20st^6$.....(5).

(2) multiplied by (3) gives $\Sigma\alpha^3\Sigma\alpha^4=-108s^7+36s^5t^2+60s^3t^4+12st^6$.....(6).

The second member of (5) multiplied by 3 equals the second member of (6) multiplied by 5.

$\therefore 3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4$.

II. Solution by ELMER SCHUYLER, Reading, Pa.

By Newton's Theorem $\dfrac{xf'(x)}{f(x)}=n+\dfrac{\Sigma\alpha}{x}+\dfrac{\Sigma\alpha^2}{x^2}+\cdots\cdots$

In the present case $f(x)\equiv x^3+qx+r$, $f'(x)=3x^2+q$, and

$$\frac{xf'(x)}{f(x)}=3+\frac{0}{x}-\frac{2q}{x^2}-\frac{3r}{x^3}+\frac{2q^2}{x^4}+\frac{5qr}{x^5}\cdots\cdots$$

Consequently $\Sigma\alpha^2=-2q$, $\Sigma\alpha^5=5qr$, $\Sigma\alpha^3=-3r$, $\Sigma\alpha^4=2q^2$, and $3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4=-30q^2r$.

III. Solution by J. SCHEFFER, Hagerstown, Md.

Since $\alpha+\beta+\gamma=0$, $\alpha\beta+\alpha\gamma+\beta\gamma=q$, $\alpha\beta\gamma=-r$, we find by squaring the first and applying the second, $\Sigma\alpha^2+2q=0$, whence $\Sigma\alpha^2=-2q$. By adding the three identities $\alpha^3+q\alpha+r=0$, $\beta^3+q\beta+r=0$, $\gamma^3+q\gamma+r=0$, we get $\Sigma\alpha^3+q\Sigma\alpha+3r=0$, or since $\Sigma\alpha=0$, $\Sigma\alpha^3=-3r$.

By adding the three identities $\alpha^4+q\alpha^2+r\alpha=0$, $\beta^4+q\beta^2+r\beta=0$, $\gamma^4+q\gamma^2+r\gamma=0$, we get $\Sigma\alpha^4+q\Sigma\alpha^2+r\Sigma\alpha=0$, whence $\Sigma\alpha^4=2q^2$.

By adding $\alpha^5+q\alpha^3+r\alpha^2=0$, $\beta^5+q\beta^3+r\beta^2=0$, $\gamma^5+q\gamma^3+r\gamma^2=0$, we get $\Sigma\alpha^5+q\Sigma\alpha^3+r\Sigma\alpha^2=0$, whence $\Sigma\alpha^5=5qr$.

$\therefore 3\Sigma\alpha^2.\Sigma\alpha^5=-30q^2r$, $5\Sigma\alpha^3.\Sigma\alpha^4=-30q^2r$, which proves the proposition.

IV. Solution by F. D. POSEY, A. B., San Mateo, Cal.

In the following, since $\Sigma\alpha=0$ the terms containing $\Sigma\alpha$ vanish.

$\Sigma\alpha^2=(\Sigma\alpha)^2-2\Sigma\alpha\beta=-2q$.

$\Sigma\alpha^3=(\Sigma\alpha^2)(\Sigma\alpha)-\Sigma\alpha^2\beta=-(\Sigma\alpha\beta)(\Sigma\alpha)+3\alpha\beta\gamma=-3r$.

$$\Sigma\alpha^4=(\Sigma\alpha^3)(\Sigma\alpha)-\Sigma\alpha^3\beta=-(\Sigma\alpha^2)(\Sigma\alpha\beta)+\Sigma\alpha^2\beta\gamma=-(\Sigma\alpha^2)(\Sigma\alpha\beta)+(\Sigma\alpha)\alpha\beta\gamma$$
$$=-(\Sigma\alpha^2)(\Sigma\alpha\beta)=2q^2.$$

$$\Sigma\alpha^5=(\Sigma\alpha^4)(\Sigma\alpha)-\Sigma\alpha^4\beta=-(\Sigma\alpha^3)(\Sigma\alpha\beta)+\Sigma\alpha^3\beta\gamma=-(\Sigma\alpha^3)(\Sigma\alpha\beta)+(\Sigma\alpha^2)\alpha\beta\gamma=5rq.$$

$$\therefore 3\Sigma\alpha^2\Sigma\alpha^5=5\Sigma\alpha^3\Sigma\alpha^4.$$

*** Solved by G. B. M. Zerr and Elmer Schuyler by computing the values of $\Sigma\alpha^2$, $\Sigma\alpha^3$, $\Sigma\alpha^4$, $\Sigma\alpha^5$ by means of the symmetric functions of the roots.

Solved by L. E. Newcomb by actual determination of the roots α, β, γ, of the given equation.

Also solved by G. W. Greenwood.

GEOMETRY.

226. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

The triangles ABC, $A'B'C'$ are in plane perspective, and the corresponding sides BC, $B'C'$, ..., cut in P, Q, R, respectively. AA', ..., cut the line PQR in $P'Q'R'$, respectively. Show that (PP', QQ', RR') is an involution range.

Solution by M. E. GRABER, A. M., Heidelberg University, Tiffin, Ohio.

$B(PP', QR)=B(P'P, Q'R')$ since they both equal the same range $(B'P, DC)$ [D being the intersection of $C'B'$ and AB].

Therefore (PP', QR) and $(P'P, Q'R')$ are equicross and (PP', QQ', RR') is an involution range by the theorem that if (AA', BC), $(A'A, B'C')$ are equicross the range (AA', BB', CC') will be an involution. [Lachlan's *Modern Pure Geometry*, page 272, Art. 426].

229. Proposed by F. D. POSEY, A.B., San Mateo, Cal., and G. W. GREENWOOD, M.A. (Oxon), Lebanon, Ill.

The solutions of problem 219 in the April number, "devise a simple geometric solution of the general quadratic equation," give the roots when they are *real*. Required a construction for the roots when they are *complex*.

Solution by F. D. POSEY, A. B., San Mateo, Cal.

Dr. L. E. Dickson reports a solution on page 93 of the April issue of the MONTHLY which holds when the roots are real.

When the circle on AB does not cut Ox the roots are complex. From the center of the circle AB let fall a perpendicular upon Ox cutting the circle at C and Ox at P. Produce this line to D making $CPD=q+1$. On CD as diameter describe a circle cutting Ox at M and N. Then the roots of the equation are $OP+PM\sqrt{(-1)}$ and $OP-PM\sqrt{(-1)}$.

Proof. CP computed in terms of p and q is found to be

$$\frac{q+1-\sqrt{[p^2+(q-1)^2]}}{2}.$$

$$\therefore PD=CD-CP=\frac{q+1+\sqrt{[p^2+(q-1)^2]}}{2}. \quad \therefore PM^2=CP.PD=-\tfrac{1}{4}p^2+q.$$

$\therefore (OP+PM_1\sqrt{-1})(OP-PM_1\sqrt{-1})=OP^2+PM^2=\frac{1}{4}p^2-\frac{1}{4}p^2+q=q$, and $(OP+PM_1\sqrt{-1})+(OP-PM_1\sqrt{-1})=2OP=-p$.

230. Proposed by SAUL EPSTEEN, Ph. D., The University of Chicago.

Cut off a given area S from a given triangle ABC by means of a line passing through a given point P, (i) when P is on a side of ABC, (ii) when P is within ABC ($S<$area ABC). [For the case P outside of ABC see problem 218].

I. Solution by J. SCHEFFER, Kee Mar College, Hagerstown, Md.

(i) The point P lies on the side AB. If S is smaller than $\triangle ACP$, find h from the proportion $AP:m=2m:h$, when $S=m^2$.

Lay off h perpendicularly to AB between the sides AC and AB. Connect the point where it strikes AC with P, and $\triangle APQ$ will be the required portion cut off. If $S>ACP$, take BP instead of AP.

(ii) Point P within the triangle.

Draw DE parallel to AB through P. Make parallelogram $LEFA=S$. On F erect perpendicular $FG=DP$, make $GQ=PE$, and connect Q with P extending to I, then will IQ be the required line. For

$$\frac{\triangle FQH}{\triangle PHE}=\frac{PE^2-DP^2}{PE^2}=1-\frac{DP^2}{PE^2}=1-\frac{\triangle DPI}{\triangle PHE}.$$

$\therefore \triangle FAH=\triangle PHE-\triangle DPI$, or $\triangle FQH+\triangle DPI=\triangle PHE$.
$\therefore \triangle FQH+\triangle DPI+AFHPD=\triangle PHE+AFHPD$.
$\therefore \triangle AQI=AFED=S$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let ABC be the given triangle altitude h, P the point on side AB, Q the point within. Lay off $AD=2S/h$ and draw BD; then $ABD=S$.

(i) Draw PD, and parallel to PD draw BM; join PM. The triangle PDM =triangle PDB.

$\therefore PDM+ADP=APM=ADP+PDB=ADB=S$.

(ii) Through Q draw HG parallel to AC. Draw HD, and parallel to HD draw BE. Join HE. Then $HED=HBD$, and $AEH=ADB$. Bisect AE at K, and draw KF parallel to AH. Then $AKFH=AEH=ADB$. At K draw KL perpendicular to AC, and make $KL=HQ$, $LM=QF$.

$\therefore KL^2+KM^2=LM^2$. $\therefore HPQ+KNM=QFN$.

$\therefore PAM$ is the required line. For $PAM=AKFH=ADB=S$.

III. Solution by G. W. GREENWOOD, M. A. (Oxon), Lebanon, Ill.

The following universal construction is Euclidean, but the proof is based on conics.

From the angle A cut *any* triangle XAY equal to the given area. Construct the triangle ZAZ' where Z and Z' lie on the sides of the angle A, and such that $AZ=AZ'=\sqrt{(AX.AY)}$. Then the area of this triangle is also equal to the given area. All lines cutting from the angle A a triangle having this area are tangent to an hyperbola having A as center and the sides of the angle as asymptotes. ZZ' is the tangent at a vertex. The circle with A as center and tangent to ZZ' is the auxiliary circle. It will cut the sides of the angle A in points which we will call M, N. At these points erect perpendiculars to the sides of the angle; these will intersect in the corresponding focus of the hyperbola S, say. Construct the circle with PS as diameter, and if this intersects the auxiliary circle, call one point of intersection Q. The angle $\angle SQP$ is right, and hence PQ is a tangent to the hyperbola, and so cuts off from the angle A the required area. There is no solution when there is no point Q, nor, according to the limitations of the question, when the segment PQ intersects the segment BC.

We can treat the angles B and C in like manner.

Also solved by Elmer Schuyler, and A. H. Holmes.

231. Proposed by B. F. FINKEL, A. M., Drury College, Springfield, Mo.

A man starts from the vertex, A, of a right isosceles triangle ABC, right-angled at A, and walks to D, the middle point of BC; from D to E, the middle point of AC; from E to F, the middle point of AD; from F to G, the middle point of DE; from G to H, the middle point of EF; from H to I, the middle point of FG; from I to J, the middle point of HI; and so on *ad infinitum*. Find the coördinates of his limiting position. [Suggested by Dr. Crawley].

I. Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

From J he walks to K, the mid-point of HI. He then performs the same journeys in the triangle KIJ as in ABC, and so on. The sides of KIJ are $\frac{1}{16}$ the length of the sides of ABC. Taking A as origin, the coördinates of K are $\frac{9}{16}AD$, and $\frac{3}{16}DC=\frac{3}{16}AD$. Hence the coördinates of his limiting position are

$$\tfrac{9}{16}AD\left(1+\frac{1}{16}+\frac{1}{16^2}+\frac{1}{16^3}+\frac{1}{16^4}+\ldots\ldots\right)=\tfrac{9}{15}AD=\tfrac{3}{5}AD,$$

$$\tfrac{3}{16}AD\left(1+\frac{1}{16}+\frac{1}{16^2}+\frac{1}{16^3}+\frac{1}{16^4}+\ldots\ldots\right)=\tfrac{3}{15}AD=\tfrac{1}{5}AD.*$$

II. Solution by A. H. HOLMES, Brunswick, Maine.

Draw the lines AD, DE, EF, FG, GH, HI, IJ, JK, and KL, according to the directions of the problem. Join AG and this line will pass through the point K. This is easily seen by drawing GN parallel to DC, meeting AD in N, and KM parallel to DC, meeting AD in M, and considering the similar triangles

*Taking D as the origin, DC and DA as the axes, the coordinates of the limiting point become $\frac{1}{5}AD$, $\frac{1}{5}AD$. This result agrees thus with the one in Solution II by Mr. Holmes. Ed.

AKM and AGN.* Similarly, a line from H to D will pass through the point L.

The point P, where AG and HD intersect is the limiting position. For every triangle within KIJ whose homologous sides are parallel to those of KIJ will have the same relation to KIJ that the latter has to ABC. Therefore the line AG passes through the right angled vertices of all such triangles within KIJ, and HD passes through the middle points of all the bases.

Let D be the origin of coördinates, DC the axis of x and DA the axis of y, then the coördinates of P are easily found to be $x=\frac{1}{3}DC$, $y=\frac{2}{3}DC$.

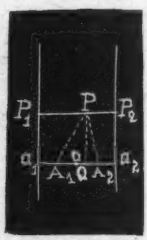
Also solved by L. E. Newcomb, Elmer Schuyler, F. D. Posey, and G. W. Greenwood.

232. Proposed by O. VEBLEN, Ph. D., The University of Chicago.

Given two parallel lines a_1, a_2, and two points A_1, A_2, upon a common perpendicular to a_1, a_2 such that A_1 is at the same distance from a_1 as A_2 is from a_2. Let P_1 be the foot of the perpendicular from a point P of the same plane to the line a_1 and P_2 the foot of the perpendicular from P to a_2. Find the locus of P when $\frac{PA_1}{PP_1}=\frac{PA_2}{PP_2}$.

Solution by J. SCHEFFER, Hagerstown, Md., and A. H. HOLMES, Brunswick, Maine.

Choosing a_1a_2, a common perpendicular to the lines a_1, a_2 for the axis of x, its middle point O for the origin of orthogonal coördinates, so that $OQ=x$, $PQ=y$, and denoting $Oa_1=Oa_2$ by a, and $OA_1=OA_2$ by b, we have $PA_1=\sqrt{[y^2+(b+x)^2]}$, $PP_1=a+x$, $PA_2=\sqrt{[y^2+(b-x)^2]}$, $PP_2=a-x$.



From the condition of the problem

$$\frac{\sqrt{[y^2+(b+x)^2]}}{a+x}=\frac{\sqrt{[y^2+(b-x)^2]}}{a-x}.$$

Squaring, clearing of fractions, and simplifying, we finally and without difficulty obtain the equation

$$\frac{y^2}{b(a-b)}+\frac{x^2}{ab}=1.$$

If $a>b$, that is, for the case that A_1 and A_2 are situated within the parallels a_1 and a_2, the equation is that of an ellipse, whose foci are A_1 and A_2, semi-axes $\sqrt{(ab)}$, and $\sqrt{[b(a-b)}$.

If $a<b$, that is, for the case that A_1 and A_2 lie outside of the parallels a_1 and a_2, the curve is an hyperbola.

Also solved by G. B. M. Zerr, L. E. Newcomb, and G. W. Greenwood.

*$AN=\frac{1}{3}AD$, $NG=\frac{1}{3}AD$; $AM=\frac{2}{5}AD$, $MK=\frac{2}{5}AD$; hence $AN/NG=AM/MK$. ED.

GROUP THEORY.

3. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Show that the equation $x^4-ax^3+bx^2-ax+1=0$, where a and b are rational numbers, while neither $2-b+(\frac{1}{2}a)^2$ nor $(1+\frac{1}{2}b)^2-a^2$ is the square of a rational number, is irreducible in the domain R of rational numbers; determine its group for this domain.

Solution by R. L. Börger, A. B., Graduate Student, The University of Chicago.

If reducible it could assume either of the forms:

$$(1)\quad (x^2+px+1)(x^2+qx+1)=x^4+(p+q)x^3+(2+pq)x^2+(p+q)x+1=0.$$

$$(2)\quad (x^2+px+r)(x^2+qx+\frac{1}{r})=x^4+(p+q)^3+(r+\frac{1}{r}+pq)x^2+(qr+\frac{p}{r})x+1=0$$

From (1), equating coefficients,

$$p+q=-a\,;\ pq=b-2.\quad \therefore \left.\begin{matrix}p\\q\end{matrix}\right\}=-\tfrac{1}{2}a\pm\sqrt{(\tfrac{1}{2}a)^2+2-b}\,.$$

From (2), $p+q=-a\,;\ qr+\dfrac{p}{r}=-a\,;\ r+\dfrac{1}{r}+pq=b.\quad \therefore p=-\dfrac{ar}{r+1}\,;\ q=-\dfrac{a}{r+1}.$

Substituting these in last we get

$$\frac{r^2+1}{r}+a^2\frac{r}{(r+1)^2}=b,\ \text{or}\ \frac{(r+1)^2}{r}+a^2\frac{r}{(r+1)^2}=b+2.\quad \text{Put}\ \frac{(r+1)^2}{r}=x.$$

Then we have,

$$x^2-(b+2)x+a^2=0,\ \text{or}\ x=\frac{b+2}{2}\pm\sqrt{\frac{(b+2)^2}{4}-a^2}\,.$$

Now if the radicals for the above two quadratics are irrational then the original equation can not be reducible. Hence the equation is irreducible if neither $(2-b)+(\frac{1}{2}a)^2$ nor $(1+\frac{1}{2}b)^2-a^2$ is the the square of a rational number.

To determine the group of the equation. Call its roots α_0, β_0, α_1, β_1 where the β's are reciprocals of the α's. Then $\alpha_0\beta_0+\alpha_1\beta_1=2$. Its conjugates under G_4 are $\alpha_1\beta_0+\alpha_0\beta_1$ and $\alpha_0\alpha_1+\beta_0\beta_1$. No two of them can be equal for then the equation would have equal roots, which is impossible because of its irreducibility. The group is then either either G_8 consisting of

$$[1,\ (\alpha_0\beta_0);\ (\alpha_1\beta_1);\ (\alpha_0\beta_0)(\alpha_1\beta_1);\ (\alpha_0\beta_1)(\alpha_1\beta_0);\ (\alpha_0\alpha_1)(\beta_0\beta_1);\ (\alpha_0\alpha_1\beta_0\beta_1);\ (\alpha_1\alpha_0\beta_1\beta_0)],\ \text{or}$$

(a) 1; $(\alpha_0\beta_0)$; $(\alpha_1\beta_1)$; $(\alpha_0\beta_0)(\alpha_1\beta_1)$,

(b) 1; $(\alpha_0\beta_0)(\alpha_1\beta_1)$; $(\alpha_0\alpha_1\beta_0\beta_1)$; $(\alpha_1\alpha_0\beta_1\beta_0)$,

(c) 1; $(\alpha_0\beta_0)(\alpha_1\beta_1)$; $(\alpha_0\alpha_1)(\beta_0\beta_1)$; $(\alpha_0\beta_1)(\alpha_1\beta_0)$.

The group must be transitive since the equation is irreducible. Hence we may exclude (a).

It cannot be (b) since (b) is the regular cyclic group, and the equation, if this were its group, would be Abelian.

We compute $(\alpha_0-\beta_0)(\alpha_1-\beta_1)$ which belongs to the group (c):

$$\alpha_0\beta_0+\alpha_1\beta_1+\alpha_0\beta_1+\alpha_1\beta_0+\alpha_0\alpha_1=b.$$

$$\therefore \alpha_0\beta_1+\alpha_1\beta_0+\alpha_0\alpha_1+\beta_0\beta_1=b-2.$$

Factoring this, $(\alpha_0+\beta_0)(\beta_1+\beta_1)=b-2$. Squaring, and remembering that $\alpha_0\beta_0=\alpha_1\beta_1=1$,

$$(\alpha_0-\beta_0)^2(\alpha_1-\beta_1)^2=[(\alpha_0+\beta_0)^2-4][(\alpha_1+\beta_1)^2-4].$$

$\therefore (\alpha_0-\beta_0)(\alpha_1-\beta_1)=2\sqrt{[(1+\frac{1}{2}b)^2-a^2]}$, and as the irrationality of the radical is one of the conditions for irreducibility this group cannot be the group of the equation. The group of the equation is therefore G_8.

PROBLEMS FOR SOLUTION.

ALGEBRA.

208. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, O.

Solve $x^4+y^4=14x^2y^2$; $x+y=m$.

209. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, O.

Prove that $(a^4+b^4+c^4+d^4)>4abcd$.

210. Proposed by W. J. GREENSTREET, A. M., Editor of The Mathematical Gazette, Stroud, England.

The sum of five quantities and the sum of their cubes are both zero. Show that the sum of their fifth powers is a factor of the sum of any odd powers of the quantities.

GEOMETRY.

236. Proposed by J. R. HITT.

If two sides of a triangle pass through a fixed point, the third side touches a fixed circle.

237. Proposed by S. A. COREY, Hiteman, Iowa.

Let AB, BC, CD, DE, EA be the sides of a pentagon, plain or gauche. Double the length of CB and DE by extending from B and E to G and H, respectively. Draw $B'D$ parallel to and of the same currency as BC. Connect G and H. Then prove that $2(AB^2+BC^2+CD^2+DE^2+EA^2)=3CD^2+4(DE.BC.\cos EDB+EA.AB.\cos EAB)+GH^2$.

238. Proposed by O. W. ANTHONY, Head of the Mathematical Department, DeWitt Clinton High School, New York.

Construct a trapezoid having given the sum of the parallel sides, the sum of the diagonals, and the angle formed by the diagonals.

CALCULUS.

183. Proposed by W. J. GREENSTREET, A. M., Stroud, England.

Evaluate $\int_0^\infty \frac{\sin 2nx\,dx}{(a^2+x^2)\sin x}$.

184. Proposed by W. J. GREENSTREET, A. M., Stroud, England.

If $u=f(x, y)$; $\xi=e^x y$; $y\eta=e^x$; show that

$$\frac{d^2u}{dx^2}-y^2\frac{d^2u}{dy^2}-y\frac{du}{dy}=4\xi\eta\frac{d^2u}{d\xi.d\eta}.$$

MECHANICS.

121. Proposed by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Prove that the electrical capacity of an oblate ellipsoid of revolution is $\sqrt{(a^2-b^2)}/\cos^{-1}(b/a)$, where a and b are the equatorial and polar semi-diameters.

AVERAGE AND PROBABILITY.

156. Proposed by J. E. SANDERS, Hackney, Ohio.

Find the average area of a triangle, the sum of whose sides is constant and equal to $2a$.

DIOPHANTINE ANALYSIS.

122. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

If p is a prime $(p^4-1(p^2-1)$ has no factor of the form $1+p^3x$, $x>0$, if $p>2$; $(p^6-1)(p^4-1)(p^2-1)$ has no factor of the form $1+p^5x$, $x>0$.

NOTES.

Professor George Pirie of the University of Aberdeen, died on August 21, at the age of sixty-one years.

Dr. L. P. Karpenski and Dr. J. W. Bradshaw have been appointed instructors in mathematics in the University of Michigan.

Dr. Ludwig Sylow, Professor of Mathematics in the University of Christiana, has been made a knight of the Prussian order 'pour le mérite.'

Dr. Moritz Cantor, of the University of Heidelberg, author of the "Geschichte der Mathematik," celebrated recently his seventy-fifth birthday.

On June 23, 1904, the University of Oxford, England, conferred the M. A. degree upon G. W. Greenwood, of Lebanon, Ill. Professor Greenwood was presented for the degree by Canon Moore of Canterbury.

Again death has taken away a valued contributor and friend of the MONTHEY in the person of Dr. George Lilley, who died June 8th, at Eugene, Ore. He had long been a sufferer from a complication which finally terminated in artero-sclerosis, the result of too close application to his chosen profession.

Dr. Lilley received his education at Knox College, Michigan University, and Wesleyan University, Illinois, the last conferring upon him the degrees of Ph. D. and LL. D. He organized and opened, as its president, the Dakota Agricultural College, and was later its professor of mathematics for four years.

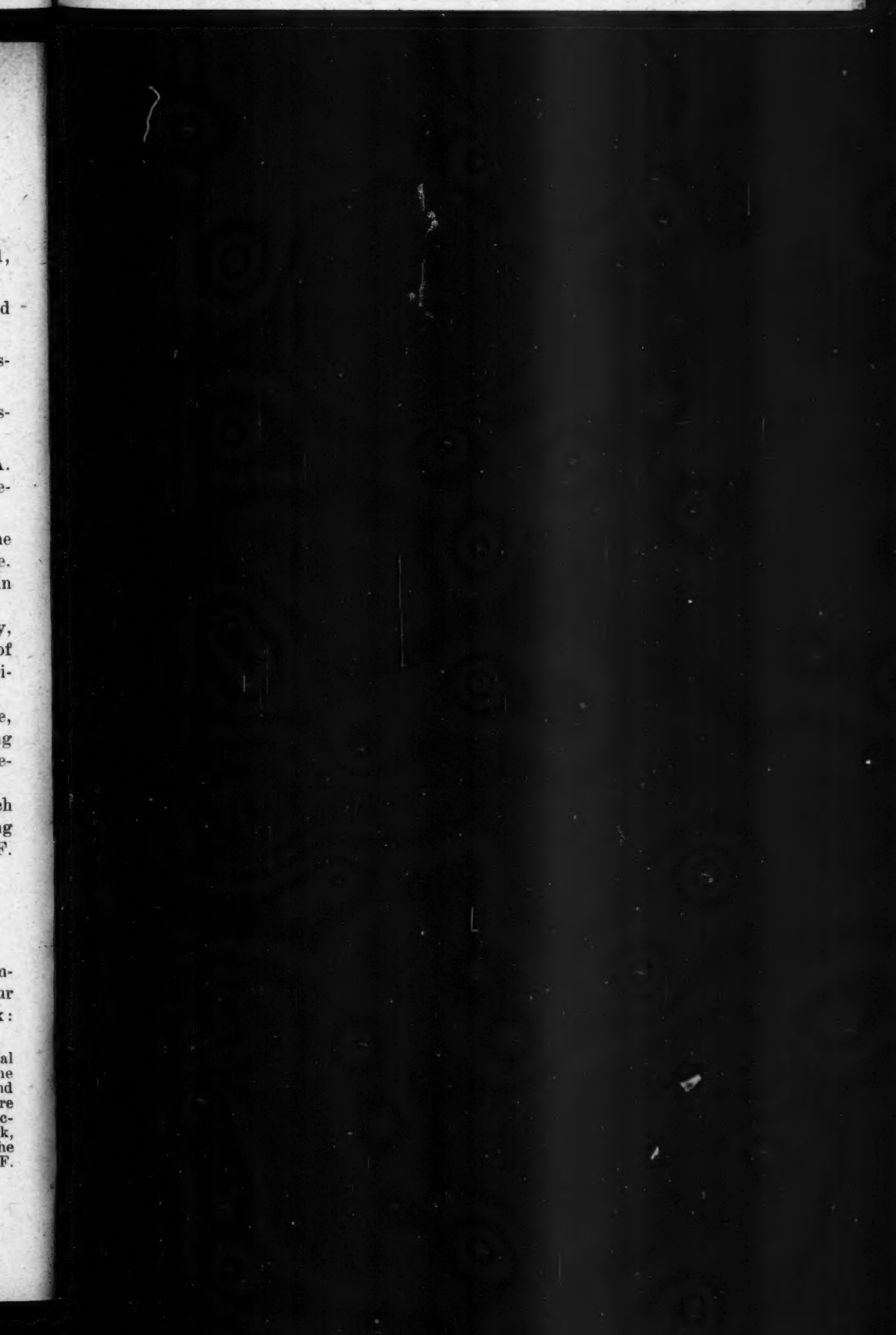
The organizing and opening of the Washington Agricultural College, Pullman, Washingtan, was his next educational achievement. After remaining at that institution two years as its president, he went to Eugene, where he retained the chair of mathematics of the University of Oregon for five years.

Professor Lilley was the author of several text books on Algebra, which have long been in use in schools. He started a treatise on Geometry, but owing to ill health was obliged to give up the work. F.

BOOKS.

The Elements of the Differential and Integral Calculus with Numerous Examples. By Donald Francis Campbell, Ph. D., Professor of Mathematics, Armour Institute of Technology. 12mo. Cloth, x + 364 pages. Price, $1.90. New York: The Macmillan Co.

While the aim of this work is primarily to meet the needs of students in technical colleges, yet it is well adapted to the requirements of the average student pursuing the subject in most colleges. The theory of definite integration is accurately explained, and two methods of approximate integration, viz., the Trapezoidal, and Simpson's, are explained. The last eighty-five pages of the book are devoted to an elementary and practical discussion of mechanics. The author has not hesitated to introduce, in this work, the important Elliptic Integrals, so noticeably omitted in many elementary works. The book is well written and the diagrams are very good. B. F. F.



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THE SUBTRACTION GROUPS.

By G. A. MILLER.

In the preceding volume of this journal* we considered the groups of the figures which are studied in elementary geometry and found that all the groups of the plane figures are of the dihedral rotation type and that every dihedral rotation group, with the exception of the one of order 4, is the group of movements of some regular polygon. In the present article we shall prove that these same groups are the groups of subtraction when the remainders are taken with respect to a modulus.

Although we shall deal with analytic concepts yet it seems best to make extensive use of geometric considerations. Let $x_1, x_2, \ldots\ldots, x_l$ denote any set of real integers and suppose that they are represented in the usual manner by points on a line. The numbers $n-x_1, n-x_2, \ldots\ldots, n-x_l$ (where n is any real number) correspond to the points obtained by reflecting the points corresponding to $x_1, x_2, \ldots\ldots, x_l$ on the point midway between 0 and n. Since reflection is an operation of period two subtraction from a single number is an operation of period two.

Instead of considering the remainders obtained by subtracting different numbers from the same number it seems desirable to consider the remainders when a given number is subtracted from a set of different numbers; *e. g.* the numbers $x_1-n, x_2-n, \ldots\ldots, x_l-n$. These l distinct operations of period two are in general, not commutative; for if we first subtract n from x_1 and then the resulting number from x_2 we obtain $n+x_2-x_1$, while $n+x_1-x_2$ is obtained when

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